Modelling with geometry: Differential geometry of curves and surfaces

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1 Introduction

Differential geometry uses the tools of multi-variate calculus (and linear algebra) to study the "geometry" of non-linear spaces. Roughly speaking, one wishes to study and understand the possible shapes of curves and surfaces in space. The main reference for this course will be John Oprea's *Differential geometry and its applications*. Let me quote from its preface:

> How and what should we teach today's undergraduates to prepare them for careers in mathematically oriented areas? Furthermore, how can we ameliorate the quantum leap from introductory calculus and linear algebra to more abstract methods in both pure and applied mathematics? There *is* a subject which can take students of mathematics to the next level of development and this subject is, at once, intuitive, calculable, useful, interdisciplinary and, *most importantly*, interesting. Of course, I'm talking here about Differential Geometry, a subject with a long, wonderful history and a subject which has found new relevance in areas ranging from machinery design to the classification of four-manifolds to the creation of theories of Nature's fundamental forces to the study of DNA.
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> Differential Geometry provides the perfect transition course to higher mathematics and its applications. It is a subject which allows students to see mathematics for what it is — not the compartmentalized courses of a standard university curriculum, but a unified whole mixing together geometry, calculus, linear algebra, differential equations, complex variables, the calculus of variations and various notions from the sciences. Moreover, Differential Geometry is not just for mathematics majors, but encompasses techniques and ideas relevant to students in engineering and the sciences. Furthermore, the subject itself is not quantized. By this, I mean that there is a continuous spectrum of results that proceeds from those which depend on calculation alone to those whose proofs are quite abstract. In this way students gradually are transformed from calculators to thinkers.

2 Prerequisities

Essential:

- Differential and integral multi-variate calculus.

- Linear algebra

Recommended:

- A first course in ordinary differential equations.

3 Course contents

- **Curves**. Curves in the plane and space; local theory; global results; applications.

  Some extra topics for mini-projects, homework or class:

  - Tait-Kneser Theorem and generalizations
  - Involutes, gears and clocks.
  - Bicycle geometry
  - The classical theorems: isoperimetric inequality, 4 vertex theorem, Crofton formula, Fary-Milnor theorem, The moon in a puddle, etc.
  - Linking number, supercoiling, White's formula.

- **Surfaces in 3-space**. Parametrized surfaces; first fundamental form (the metric); examples; Gauss map, curvature; intrinsic geometry, parallel transport and geodesics; global results.

  Some extra topics for mini-projects, homework or class:

  - Ruled and minimal surfaces
  - Delaunay surfaces
  - Gauss-Bonnet, classification of compact surfaces
  - The hyperbolic plane.

- **Applications:** a few applications contained in the above themes plus an introduction to quaternions and their use as 3-space rotations.

4 Grading

There will be weekly homework counting between 30% and 40% of the final grade. If few students are enrolled (≤ 8), there will be more weight on projects, but at least we'll have one exam. After the first week of class I'll be more precise about grading. Attendance is mandatory.

5 References

- *Differential geometry and its applications*, J. Oprea.
- *What is differential geometry?*, A. Petrunin & S. Zamora.
- *Differential Geometry: A First Course in Curves and Surfaces*, T. Shifrin.
- *Lecture Notes on Differential Geometry*, M. Ghomi.
- *Modern differential geometry of curves and surfaces with Mathematica*, A. Gray.

6 A note about the lecturer's teaching experience

The lecturer finished a degree in mathematics from UNAM and obtained his Ph.D at the University of Utah. While at Slat Lake City, he taught several undergraduate courses ranging from elementary math to calculus and linear algebra. After that, he was an L.E. Dickson Instructor in Mathematics, at the University of Chicago, where he taught advanced undergraduate and graduate courses, in geometry and analysis. For two years he was an Asistant Professor at Ohio State University, teaching a wide range of undergraduate courses including several honors' calculus courses. Since 1994 he's been a researcher at Cimat teaching undergraduate, graduate and highschool courses in all areas of mathematics. He's also been a visiting professor to U. of Santiago de Compostela (several timnes) where he's taught graduate courses in topology and geometry. He's currently a member of the National Researchers System and the Mexican Academy of Sciences.